$$P_{ij}(s) \triangleq \sum_{n=0}^{\infty} \underbrace{P_{ij}(n)}_{\mathbb{P}(X_n=j \mid X_0=i)} \cdot s^n ,$$

$$\begin{cases} P_{ii}(s) = 1 + F_{ii}(s) P_{ii}(s) \\ P_{ij}(s) = F_{ij}(s) \cdot P_{jj}(s) \\ \qquad (\forall i \neq j) \end{cases}$$

$$F_{ij}(s) \triangleq \sum_{n=0}^{\infty} \underbrace{f_{ij}(n)}_{\mathbb{P}(T_j = n \mid X_0 = i)} \cdot s^n$$

State y $\begin{cases} \text{recurrent} & \mathbb{P}_y(T_y < \infty) = 1 \\ \text{transient} & \mathbb{P}_y(T_y < \infty) < 1 \end{cases}$

with $\mathbb{P}_y(T_y < \infty) = F_{yy}(1)$

So: y is recurrent iff $F_{yy}(1) = 1$

iff $$\frac{P_{yy}(1) - 1}{P_{yy}(1)} = 1$$

iff $P_{yy}(1) = \infty$

iff $$\sum_{n=0}^{\infty} P_{yy}(n) = \infty$$

(6.2.3)

e.g.: State y is recurrent for Markov chain $\{X_n\}$ iff the expected # of visits to y is ∞ if starts from $X_0 = y$.

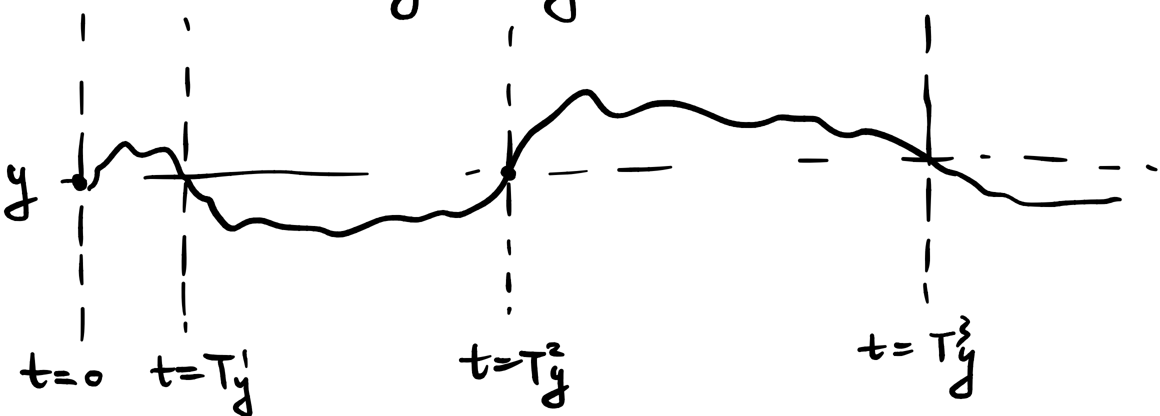
Pf: $N_y \triangleq \sum_{n=0}^{\infty} I_{\{X_n = y\}}$ is the # of visits to y

$$\mathbb{E}_y N_y = \mathbb{E}_y \sum_{n=0}^{\infty} I_{\{X_n=y\}} \overset{\text{Fubini}}{=\!=} \sum_{n=0}^{\infty} \mathbb{P}_y(X_n = y)$$

$$= \sum_{n=0}^{\infty} P_{yy}(n) = \infty$$

iff y is recurrent.

Intuition: if $\mathbb{P}_y(T_y < \infty) = 1$, then restart MC once it hits y, expect to see infinitely many hits.



Understanding recurrent/transient intuitively:

e.g.: If RW $\{S_n\}$ has i.i.d. increments $\{X_n\}$, $\mathbb{E}|X_1| < \infty$, $\mathbb{E}X_1 \neq 0$, then state 0 is transient.

Pf: SLLN: $\frac{S_n}{n} \xrightarrow{a.s.} \mathbb{E}X_1 \quad (n \to \infty)$

WLOG, assume $\mathbb{E}X_1 > 0$, so $S_n \xrightarrow{a.s.} +\infty \; (n \to \infty)$.

With prob 1, $\exists\, K > 0$, $\forall\, n \geq K$, $S_n \geq 1$.

So: $\mathbb{P}_0(N_0 \leq K) = 1$, $\mathbb{E}_0 N_0 \leq K < \infty$,

State 0 must be transient.

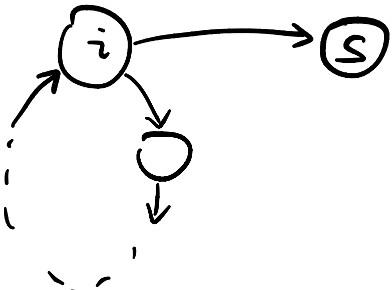
obvious "trend" $\longrightarrow$ transience

(e.g. asymm SRW, absorbing states, etc.)

e.g. (6.2.2) $\{X_n\}$ has absorbing state s and s communicate with any other states,

$\forall i \in S$, $\exists n = n(i)$, $P_{is}(n) > 0$. Show that all states other than s are transient.

Pf: $\forall i \in S$, $n_i \triangleq \min\{n : P_{is}(n) > 0\} < \infty$



Consider two procedures: $\begin{cases} i \to s \\ i \to i \end{cases}$,

let $T_i^k \triangleq$ the k-th time Markov chain hits i (except time 0)

then $(0, T_i^1)$, (T_i^1, T_i^2), ---, (T_i^{k-1}, T_i^k), -- are visits to state i

If i is recurrent, by contradiction,

$$\mathbb{P}_i(T_i^k < \infty) = \mathbb{P}_i(T_i^{k-1} < \infty, T_i^k < \infty)$$

$$= \mathbb{P}_i(T_i^{k-1} < \infty) \cdot \mathbb{P}_i(T_i^k < \infty \mid T_i^{k-1} < \infty)$$

Strong Markov

$$= \mathbb{P}_i(T_i^{k-1} < \infty) \cdot \mathbb{P}_i(T_i^1 < \infty)$$

As a result, $\mathbb{P}_i(T_i^k < \infty) = \left[\mathbb{P}_i(T_i^1 < \infty)\right]^k$.

By recurrence of i, $\forall k$, $\mathbb{P}_i(T_i^k < \infty) = 1$

if a state is recurrent, with prob 1, the k-th hitting time is finite for $\forall k$

★ Not only expected # of hitting is ∞

$$\mathbb{E}_i N_i = \infty$$

the # of hitting is almost surely ∞

$$\mathbb{P}_i(N_i = \infty) = 1.$$

However, $\mathbb{P}_i(X_{n_i} = s) > 0$ and whenever it hits s, it permanently stays at s, so

$$\mathbb{P}_i(T_i^{n_i} = \infty) \geq \mathbb{P}_i(X_{n_i} = s) > 0,$$

Contradiction!

Classification of states:

Irreducible: one communication class

Closed: never transit out of the state subset

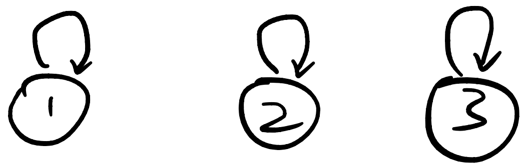
- Communication class share recurrence/transiensce
- Finite closed set $C \subseteq S$ contains positive recurrent state

recurrence
- null recurrent $(\mathbb{E}_i T_i = \infty)$
- positive recurrent $(\mathbb{E}_i T_i < \infty)$

e.g: (6.3.3)

$$P = \begin{pmatrix} 1-2p & 2p & 0 \\ p & 1-2p & p \\ 0 & 2p & 1-2p \end{pmatrix}, \quad p \in [0, \tfrac{1}{2}]$$

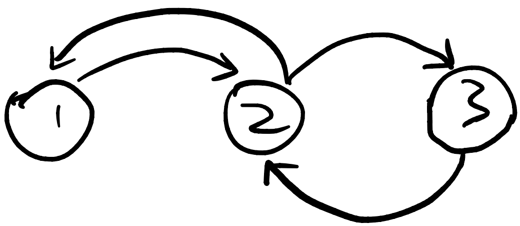
①: $p = 0$



all positive recurrent

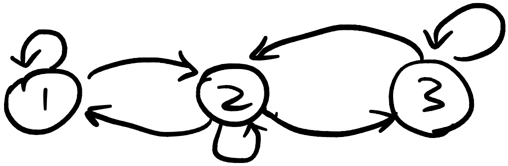
mean recurrence time $= 1$

②: $p = \frac{1}{2}$



Irreducible, finite state $\Rightarrow$ all positive recurrent

③: $0 < p < \frac{1}{2}$



all positive recurrent

Calculate mean recurrence time?

first step decomposition

$$\mathbb{E}_1 T_1 = \mathbb{P}_1(X_1 = 1)\cdot \mathbb{E}_1(T_1 \mid X_1 = 1) + \mathbb{P}_1(X_1=2)\cdot \mathbb{E}_1(T_1 \mid X_1 = 2)$$

$$\overset{\text{Markov}}{=} (1-2p)\cdot 1 + 2p\cdot(1 + \mathbb{E}_2 T_1)$$

$$= 1 + 2p\cdot \mathbb{E}_2 T_1$$

$$\mathbb{E}_2 T_1 = \mathbb{P}_2(X_1 = 1)\cdot \mathbb{E}_2(T_1 \mid X_1 = 1) + \mathbb{P}_2(X_1 = 2)\cdot \mathbb{E}_2(T_1 \mid X_1 = 2) + \mathbb{P}_2(X_1 = 3)\cdot \mathbb{E}_2(T_1 \mid X_1 = 3)$$

$$= p\cdot 1 + (1-2p)\cdot(1 + \mathbb{E}_2 T_1) + p\cdot(1 + \mathbb{E}_3 T_1)$$

$$= 1 + (1-2p)\cdot \mathbb{E}_2 T_1 + p\cdot \mathbb{E}_3 T_1$$

$$\mathbb{E}_3 T_1 = 2p\cdot(1 + \mathbb{E}_2 T_1) + (1-2p)(1 + \mathbb{E}_3 T_1)$$

$$\Rightarrow \begin{cases} \mathbb{E}_1 T_1 = 1 + 2p\,\mathbb{E}_2 T_1 \\ 2p\,\mathbb{E}_2 T_1 = 1 + p\,\mathbb{E}_3 T_1 \\ 2p\,\mathbb{E}_3 T_1 = 1 + 2p\,\mathbb{E}_2 T_1 \end{cases} \Rightarrow \begin{cases} \boxed{\mathbb{E}_1 T_1 = 4} \\ \mathbb{E}_2 T_1 = \frac{3}{2p} \\ \mathbb{E}_3 T_1 = \frac{2}{p} \end{cases}$$

mean rec time for state 1

Easier way to calculate mean rec time?

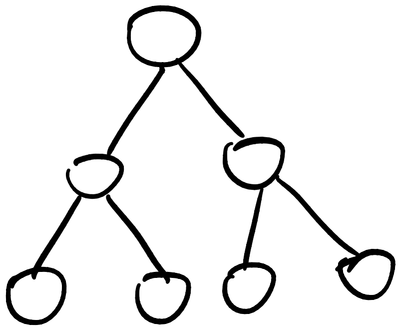
Relies on stationary dist π!

$$\mathbb{E}_i T_i = \frac{1}{\pi_i}$$

Remarkable Result

Talk about next week.

e.g: SRW on binary trees



- finite tree — all positive recurrent
- infinite tree — all transient